DYNAMICS OF AXISYMMETRIC SLENDER LIQUID BRIDGES IN ISOROTATION

D. Rivas and J. M. Vega

E.T.S.I. Aeronáuticos, Universidad Politécnica de Madrid, 28040 Madrid, Spain

ABSTRACT

The dynamics of axisymmetric slender liquid bridges in isorotation under μg conditions is studied. Slendernesses of the bridge very close to the static stability limit for cylindrical bridges are considered. The inviscid three-dimensional problem is solved in the neighborhood of that stability limit following a regular perturbation approach. A nonlinear model where the pressure and velocity fields are coupled with the free-surface deformation is obtained. The free-surface deformation in the case of stable configurations is analyzed. The effect of Coriolis forces in the dynamics of the bridge turns out to be very important: for large values of the rotation speed the movement of the free surface is very irregular, and its amplitude is much smaller than that in the case of small rotation.

INTRODUCTION

The study of the fluid configuration known as "liquid bridge" has great interest as a model of the "floating zone" technique for crystal growth. In this technique, in order to make the temperature field uniform enough, the sample (a cylindrical rod) is rotated around its axis /1/.

Even though rotation is part of the technique, in the literature the studies on liquid bridges where it is considered refer only to the static problem (including linear stability analyses) /2-4/. In this work we study the effect of the rotation of the disks (bridge supports) on the dynamics of the liquid bridge; we formulate a three-dimensional problem where the combined effect of Coriolis and capillary forces, and the nonlinear terms in the equations of motion is analyzed.

The liquid bridge will be supposed to be under microgravity conditions, thus we will consider slender bridges, that is bridges with slenderness (length/diameter ratio) close to π (the static stability limit for cylindrical bridges /5/). In practice the viscosity of the liquids commonly used is low (the appropriate dimensionless parameter being much less than one), then, in our model the viscous terms in the formulation will be neglected /6/. Finally, we will confine ourselves to the study of the axisymmetric case.

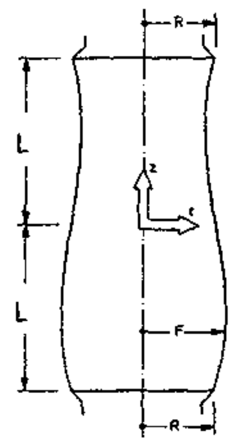

Fig. 1. Liquid bridge

PROBLEM FORMULATION

Let us consider a liquid bridge consisting of two disks of radius R separated a distance $2L$, and the liquid of density ρ and surface tension σ, as shown in figure 1. We consider a coordinate system rotating with angular velocity Ω (the rotating speed of both disks). The dimensionless, inviscid, axisymmetric equations of motion and boundary conditions in cylindrical coordinates are

$$u_r + \frac{u}{r} + w_z = 0 \tag{1a}$$

$$u_t + uu_r + wu_z - \frac{v^2}{r} = -q_r + W_e r + 2\sqrt{W_e}v \tag{1b}$$

$$v_t + uv_r + wv_z + \frac{uv}{r} = -2\sqrt{W_e}u \tag{1c}$$

$$w_t + uw_r + ww_z = -q_z \tag{1d}$$

$$@z = \pm\Lambda : \; w = 0; \quad @r = 0 : \; u = v = w_r = 0 \tag{1e}$$

$$@r = f(z,t) \begin{cases} u = f_t + f_z w \\ q + \dfrac{ff_{zz} - 1 - f_z^2}{f(1+f_z^2)^{3/2}} = 0 \end{cases} \tag{1f}$$

$$f(\pm\Lambda, t) = 1, \quad \int_{-\Lambda}^{\Lambda} f^2 dz = 2\Lambda \tag{1g}$$

These equations must be solved with appropriate initial conditions. In equations (1) r and z are the radial and axial coordinates, u, v and w the radial, azimuthal and axial velocity components, q the pressure, t the time and $r = f(z,t)$ the equation of the free surface. The problem depends on two parameters

$$\Lambda = L/R \equiv \text{slenderness}$$

$$W_e = \rho\Omega^2 R^3/\sigma \equiv \text{Weber number}$$

Typically $0 < \Lambda < \pi$ and $0 < W_e < 10^{-2}$ (the rotation speeds are usually small /1/).

MODEL FOR SLENDER, SLOWLY ROTATING LIQUID BRIDGES

We consider the following distinguished limit (neighborhood of $\Lambda = \pi$ /3/)

$$\Lambda = \pi(1 + \varepsilon^2 \ell), \qquad W_e = 2\varepsilon^2 b \tag{2}$$

with $|\ell| \sim b \sim 1$ and $0 < \varepsilon \ll 1$, and look for solutions of the form

$$u = \varepsilon^2 u_1(r,z,\tau) + \dots, \qquad v = \varepsilon^2 v_1(r,z,\tau) + \dots, \tag{3a}$$

$$w = \varepsilon^2 w_1(r,z,\tau) + \dots, \qquad q = 1 + \varepsilon^2[b(r^2 - 1) + q_1(\tau)] + \varepsilon^3 q_2(r,z,\tau) + \dots, \tag{3b}$$

$$f = 1 + \varepsilon a(\tau)\sin z + \varepsilon^2 f_1(z,\tau) + \varepsilon^3 f_2(z,\tau) + \dots, \tag{3c}$$

in terms of the slow time variable

$$\tau = \varepsilon t. \tag{3d}$$

Introducing (2) and (3) into (1), and eliminating secular terms in the time scale $t \sim 1$ and in the orders $\mathcal{O}(\varepsilon^2)$ for the velocity field and $\mathcal{O}(\varepsilon^3)$ for the pressure field, one obtains the evolution equations for u_1, v_1, w_1, q_2 and a. Eliminating u_1, v_1 and w_1 in such equations, the following model, which only depends on the pressure q_2 and the free-surface amplitude a, is obtained

$$\frac{\partial^2}{\partial\tau^2}\left(q_{2rr} + r^{-1}q_{2r} + q_{2zz}\right) + 8bq_{2zz} = 0 \qquad \text{in } \; 0 < r < 1, \quad -\pi < z < \pi, \tag{4a}$$

$$q_{2r} = 0 \quad \text{in } r = 0; \qquad q_{2r\tau} = -(a''' + 8ba')\sin z \quad \text{in } r = 1 \tag{4b}$$

$$q_{2z} = 0 \quad \text{in } z = \pm\pi; \quad a(l + b) + \frac{3a^3}{4} + (2\pi)^{-1}\int_{-\pi}^{\pi} q_2(1,z,\tau)\sin z dz = 0, \tag{4c}$$

For simplicity we consider the case where the initial velocity field is identically null ($u = v = w = 0$); in this case the following initial conditions are obtained

$$a = a_0, \quad a' = 0, \quad q_2 = \frac{\pi^2}{4N}\left(l + b + \frac{3a_0^2}{4}\right)a_0\varphi, \quad q_{2\tau} = 0, \tag{4d}$$

where the constant N is given below, and $\varphi(r, z)$ is the unique solution of

$$\Delta\varphi = 0 \quad \text{in} \quad 0 < r < 1, \quad -\pi < z < \pi, \tag{5a}$$

$$\varphi_z = 0 \quad \text{in} \quad z = \pm\pi, \quad \varphi_r = -\sin z \quad \text{in} \quad r = 1, \quad \varphi_r = 0 \quad \text{in} \quad r = 0. \tag{5b}$$

The linear equation (4a) is known as Poincaré or Sobolev equation; the only nonlinearity appears in (4c) in the variable a.

RESULTS

The unique solution of equations (4) can be written as

$$q_2 = [a'' + 8b(a - a_0)]\varphi + 8b(a - a_0)\psi + \sum_{m=0}^{\infty} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} Q_{mn}(\tau) J_0(\alpha_m r)\cos[n(z + \pi)/2], \tag{6}$$

where $\psi(r, z)$ is the unique solution of

$$\Delta\psi = -\varphi_{zz} \quad \text{in} \quad 0 < r < 1, \quad -\pi < z < \pi, \tag{7a}$$

$$\psi_z = 0 \quad \text{in} \quad z = \pm\pi, \quad \psi_r = 0 \text{ in } r = 0, 1. \tag{7b}$$

After substituting (6) into (4) one obtains the following set of equations for Q_{mn} and a

$$a'' = -8b\left(1 - \frac{N_1}{N}\right)(a - a_0) + \frac{\pi^2}{4N}\left[(\ell + b)a + \frac{3a^3}{4}\right] - \frac{\pi}{N}\sum_{m=0}^{\infty}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{J_0(\alpha_m)}{4 - n^2} Q_{mn}, \tag{8a}$$

$$Q''_{mn} + 8bK_{mn}Q_{mn} = (8b)^2(a - a_0)C_{mn}, \quad m \geq 0, \ n \geq 1, \ n = \text{ odd}, \tag{8b}$$

$$a(0) - a_0 = a'(0) = Q_{mn}(0) = Q'_{mn}(0) = 0, \tag{8c}$$

where $\alpha_0, \alpha_1, \dots$, are the successive roots of $J_1(\alpha_m) = 0$ and

$$K_{mn} = \frac{n^2}{4\alpha_m^2 + n^2}, \qquad C_{mn} = -\frac{256\alpha_m^2 K_{mn}}{\pi J_0(\alpha_m)(4 - n^2)(4\alpha_m^2 + n^2)^2}, \tag{9a}$$

$$N = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{16 I_0(n/2)}{n I_1(n/2)(4 - n^2)^2} \simeq 7.7003, \qquad N_1 = \sum_{m=0}^{\infty}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{64K_{mn}^2}{n^2(4 - n^2)^2} \simeq 7.4111, \tag{9b}$$

J_0, J_1, I_0 and I_1 being the two first Bessel and modified Bessel functions.

The system of equations (8) has been integrated using a Runge-Kutta method of 4^{th} order, after truncating the infinite sum, typically, for $m, n \leq 100$, and using a time step $\Delta\tau = 0.001$. In figure 2 we present the free-surface amplitude $a(\tau)$ for $a_0 = 1$, $\ell = -50$ and $b = 1$, 10, 30 and 40.

In the absence of rotation (if $b = 0$) one has $Q_{mn} = 0$ for all m and n, and equation (8a) for $a(\tau)$ reduces to a subcritical Duffin equation with periodic solutions for $\ell < 0$ and $|a_0| < \sqrt{-4\ell/3}$, and divergent solutions in finite time (which correspond to the breakage of the bridge) for $\ell > 0$ or $\ell < 0$ and $|a_0| > \sqrt{-4\ell/3}$.

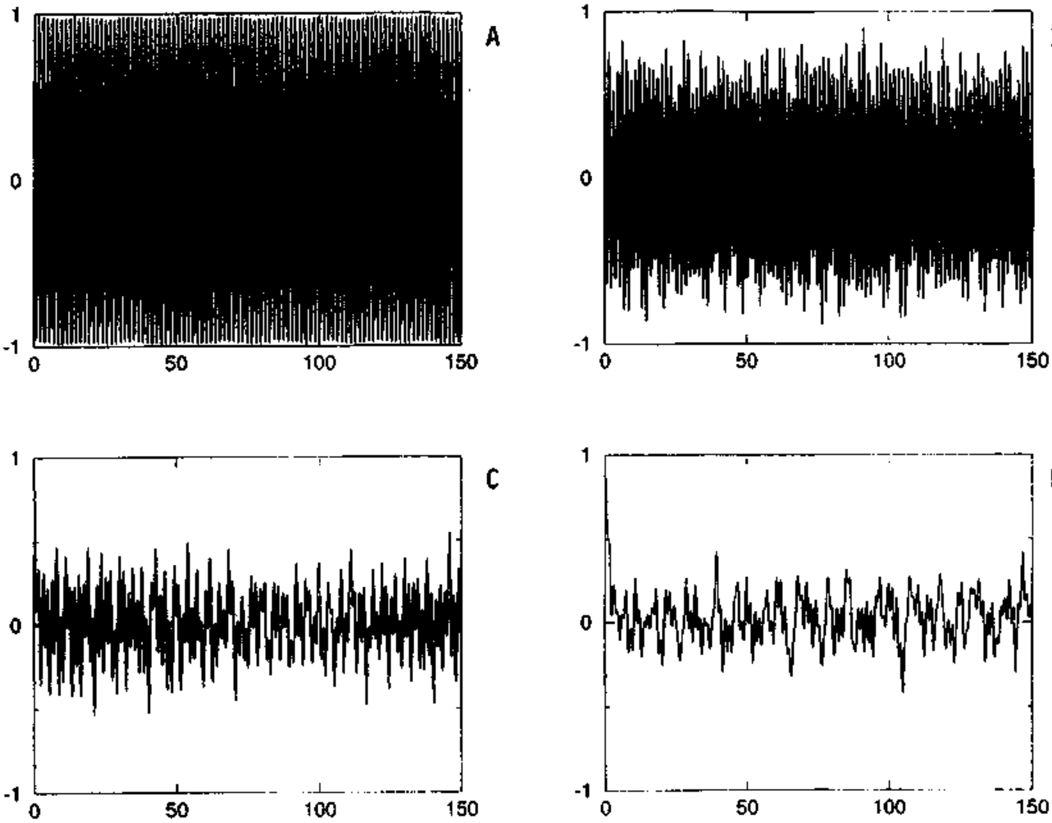

Fig. 2. Free-surface amplitude $a(\tau)$. $a_0 = 1$, $\ell = -50$. (A) $b = 1$, (B) $b = 10$, (C) $b = 30$, (D) $b = 40$.

If $b > 0$ one has the following behavior: quasi-periodic solutions (see figure 2) for $\ell + b < 0$ and $|a_0| < \sqrt{-4(\ell + b)/3}$, and divergent solutions in finite time (breakage) for $\ell + b > 0$ or $\ell + b < 0$ and $|a_0| > \sqrt{-4(\ell + b)/3}$. The quasi-periodic solutions are obtained after a transient interval, and then the local maxima of the function $|a| = |a(\tau)|$ are always clearly smaller than $|a_0| = |a(0)|$; the larger $\ell + b$ (for given a_0), the smaller those maxima. In these latter cases the free surface behavior, although irregular (see figure 2d), does not seem to be chaotic; this is concluded after calculating the Lyapunov exponents and the Fourier transforms of the function $a = a(\tau)$.

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